Supersymmetry breaking in noncommutative quantum mechanics

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Abstract
Supersymmetric quantum mechanics is formulated on a two-dimensional noncommutative plane and applied to the supersymmetric harmonic oscillator. We find that the ordinary commutative supersymmetry is partially broken and only half of the number of supercharges are conserved. It is argued that this breaking is closely related to the breaking of time reversal symmetry arising from noncommutativity.

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1. Introduction

Ever since the realization that a consistent quantum description of gravity may require a drastic change in our notion of spacetime at short lengthscales [1–3], noncommutative geometry, and particularly the formulation of quantum mechanics and quantum field theories on noncommutative spaces [4], has become a fruitful line of investigation as a possible candidate to replace our current notion of spacetime [5].

An interesting feature of noncommutative quantum mechanics that emerged from these studies is the breaking of time reversal symmetry in the presence of a non-constant potential [6, 7]. As this clearly lifts some of the degeneracies in the spectrum, it is natural to ask how this may effect other symmetries and in particular supersymmetry. This is a particularly pertinent question in the light of the findings of [8] where it was concluded that supersymmetry is half broken in the presence of noncommutativity.

The simplest setting to discuss supersymmetric quantum mechanics is in the context of the ordinary supersymmetric factorization as discussed in [9] and references therein. Here,

our aim is to generalize this supersymmetric factorization to the noncommutative case and to investigate the implications that noncommutativity has for supersymmetry.

As the prime example of a factorizable potential is the harmonic oscillator, this is also the most natural example to which the resulting formalism can be applied. In contrast to the noncommutative harmonic oscillator on which a considerable body of literature exists (see e.g. [10–12]), the noncommutative supersymmetric harmonic oscillator only received some attention recently [13, 14]. Here we follow a different approach, based on [15], and rather focus on the issue of supersymmetry breaking, which was not discussed in these papers.

Not unexpectedly the supersymmetric noncommutative harmonic oscillator Hamiltonian can be diagonalized and the quantum supercharges identified. We find that the noncommutativity partially breaks the ordinary $\mathcal{N} = 4$ supersymmetry down to $\mathcal{N} = 2$. It is argued that this breaking is directly related to the breaking of time reversal symmetry.

The paper is organized as follows. The following section is devoted to classical and quantum algebraic considerations induced by the noncommutativity. Section 3 develops our main results on noncommutative supersymmetric factorization which we apply to a solvable case, namely the harmonic oscillator in two dimensions. Section 4 discusses the question of BPS states [16] in the present context. The paper ends with some discussion in section 5.

2. Classical and quantum algebras

Before studying the supersymmetric version of the noncommutative harmonic oscillator, this section discusses some algebraic preliminaries pertaining to noncommutative coordinate algebras and their representation, thus fixing the notations of the following sections.

Noncommutative two-dimensional space is defined by the following commutation relation between coordinates:

$$[\hat{x}, \hat{y}] = i\theta. \tag{1}$$

The parameter θ will be referred to as the noncommutative parameter and has the dimension of a length squared. More generally, in the $2N$-dimensional case the commutation relations can be brought into a canonical form $[x^i, y^j] = i\Theta^{ij}$, where the antisymmetric tensor Θ^{ij} has the block diagonal form $\Theta^{ij} = \mathrm{diag}(J^1, J^2, \ldots, J^N)$ with J^j given by

$$J^j = \begin{pmatrix} 0 & \theta^j \\ -\theta^j & 0 \end{pmatrix}. \tag{2}$$

Thus, for each symplectic pair (x^j, y^j), the noncommutative parameter is $\theta^j > 0$, $j = 1, 2, \ldots, N$.

Introducing the pair of boson annihilation and creation operators $b = (1/\sqrt{2\theta})(\hat{x} + i\hat{y})$ and $b^\dagger = (1/\sqrt{2\theta})(\hat{x} - i\hat{y})$, which satisfy the Heisenberg–Fock algebra $[b, b^\dagger] = \mathbb{1}$, noncommutative configuration space is itself a Hilbert space, which we denote by $\mathcal{H}_c$, isomorphic to boson Fock space $\mathcal{H}_c = \mathrm{span}\{|n\rangle, n \in \mathbb{N}\}$, with $|n\rangle = (1/\sqrt{n!})(b^\dagger)^n|0\rangle$. In the $2N$-dimensional case classical configuration space is simply the N tensorial product of Fock space.

On the quantum level the Hilbert space in which the states of the system are represented, and which we denote by $\mathcal{H}_q$, is defined to be the space of Hilbert–Schmidt operators on $\mathcal{H}_c$ [17]:

$$\mathcal{H}_q = \{\psi(\hat{x}_1, \hat{x}_2) : \psi(\hat{x}_1, \hat{x}_2) \in \mathcal{B}(\mathcal{H}_c), \mathrm{tr}_c(\psi(\hat{x}_1, \hat{x}_2)^\dagger \psi(\hat{x}_1, \hat{x}_2)) < \infty\}, \tag{3}$$

where tr_c denotes the trace over $\mathcal{H}_c$ and $\mathcal{B}(\mathcal{H}_c)$ is the set of bounded operators on $\mathcal{H}_c$.

Next, we seek a representation on $\mathcal{H}_q$ of the noncommutative Heisenberg algebra

$$
\begin{aligned}
&[\hat{X}, \hat{Y}] = i\,\theta, \\
&[\hat{X}, \hat{P}_X] = i\,\hbar = [\hat{Y}, \hat{P}_Y], \\
&[\hat{P}_X, \hat{P}_Y] = 0.
\end{aligned}
\tag{4}
$$

Henceforth capital letters are reserved to refer to quantum operators acting on $\mathcal{H}_q$ in order to distinguish them from operators acting on noncommutative configuration space $\mathcal{H}_c$. It is easily verified that a representation is provided by the following:

$$
\hat{X}\psi = \hat{x}\psi, \qquad \hat{Y}\psi = \hat{y}\psi, \qquad \hat{P}_X\psi = \frac{\hbar}{\theta}[\hat{y}, \psi], \qquad \hat{P}_Y\psi = -\frac{\hbar}{\theta}[\hat{x}, \psi].
\tag{5}
$$

Furthermore, it can also be shown that these operators are self-adjoint with respect to the quantum Hilbert space inner product $(\phi|\psi) = \mathrm{tr}_c(\phi^\dagger \psi)$, which makes this a unitary representation.

3. Supersymmetric noncommutative harmonic oscillator

With the formal structure of the noncommutative quantum theory settled, we proceed to introduce the concept of supersymmetric factorization in a noncommutative space after which we apply the resulting formalism to the noncommutative harmonic oscillator.

3.1. Noncommutative supersymmetric factorization

Supersymmetric factorization in commutative quantum mechanics and in two or more dimensions has recently been considered in [15]. Here we generalize this approach to a noncommutative quantum system focusing on two dimensions.

We consider the following noncommutative Hamiltonian:

$$
H = \frac{1}{2m}\left(\hat{P}_X^2 + \hat{P}_Y^2\right) + V_1(\hat{X}, \hat{Y}),
\tag{6}
$$

where $V_1(\hat{x}, \hat{y})$, considered as an operator acting on configuration space, is Hermitian, which is the analogue of a real potential in commutative space. The factorized quantum Hamiltonian assumes the general form

$$
H_1 = \hbar\omega\left(B_X^\dagger B_X + B_Y^\dagger B_Y\right),
\tag{7}
$$

where the dimensionless operators B_X, $B_X^\dagger$, B_Y, $B_Y^\dagger$ are defined as

$$
B_X = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{i}{\sqrt{2m}}\hat{P}_X + W_X(\hat{X}, \hat{Y})\right), \qquad
B_X^\dagger = \frac{1}{\sqrt{\hbar\omega}}\left(-\frac{i}{\sqrt{2m}}\hat{P}_X + W_X(\hat{X}, \hat{Y})\right),
\tag{8}
$$

$$
B_Y = \frac{1}{\sqrt{\hbar\omega}}\left(\frac{i}{\sqrt{2m}}\hat{P}_Y + W_Y(\hat{X}, \hat{Y})\right), \qquad
B_Y^\dagger = \frac{1}{\sqrt{\hbar\omega}}\left(-\frac{i}{\sqrt{2m}}\hat{P}_Y + W_Y(\hat{X}, \hat{Y})\right),
\tag{9}
$$

and the Hermitian superpotentials $W_X^\dagger(\hat{X}, \hat{Y}) = W_X(\hat{X}, \hat{Y})$ and $W_Y^\dagger(\hat{X}, \hat{Y}) = W_Y(\hat{X}, \hat{Y})$ are yet to be specified. From (7) one obtains the noncommutative version of the Riccati equation in two dimensions

$$
V_1(\hat{X}, \hat{Y}) = \frac{i}{\sqrt{2m}}((\hat{P}_X W_X)(\hat{X}, \hat{Y}) + (\hat{P}_Y W_Y)(\hat{X}, \hat{Y})) + (W_X(\hat{X}, \hat{Y}))^2 + (W_Y(\hat{X}, \hat{Y}))^2, \tag{10}
$$

which is closely related to the so-called algebraic (matrix) Riccati equation. One notes that the two superpotentials W_X and W_Y are coupled. The ground-state operator Ψ_{00} should be

annihilated by both of the annihilators B_X and B_Y, but in contrast to the commutative case the solutions W_X and W_Y of the above equation cannot be written explicitly in terms of the ground state, but are given by an algebraic system

$$\frac{i\hbar}{\sqrt{2m\theta}}[\hat{Y}, \Psi_{00}] + W_X(\hat{X}, \hat{Y})\Psi_{00} = 0, \tag{11}$$

$$\frac{-i\hbar}{\sqrt{2m\theta}}[\hat{X}, \Psi_{00}] + W_Y(\hat{X}, \hat{Y})\Psi_{00} = 0. \tag{12}$$

As is customarily the case in one-dimensional supersymmetry quantum mechanics, a different Hamiltonian can be obtained by reversing the order of the operators. Here, apart from H_1, we get mixed types of operators [15]

$$H'_{11} = \hbar\omega\big(B_X B_X^\dagger + B_Y^\dagger B_Y\big), \qquad H'_{22} = \hbar\omega\big(B_X^\dagger B_X + B_Y B_Y^\dagger\big), \tag{13}$$

$$H'_{12} = \hbar\omega\big(B_Y B_X^\dagger - B_X^\dagger B_Y\big), \qquad H'_{21} = \hbar\omega\big(B_X B_Y^\dagger - B_Y^\dagger B_X\big), \tag{14}$$

$$H_2 = \hbar\omega\big(B_X B_X^\dagger + B_Y B_Y^\dagger\big), \tag{15}$$

with $H'^{\dagger}_{12} = H'_{21}$. The corresponding superpartner potentials can also be derived

$$V'_1(\hat{X}, \hat{Y}) = \frac{-i}{\sqrt{2m}}[(\hat{P}_X W_X)(\hat{X}, \hat{Y}) - (\hat{P}_Y W_Y)(\hat{X}, \hat{Y})] + (W_X(\hat{X}, \hat{Y}))^2 + (W_Y(\hat{X}, \hat{Y}))^2, \tag{16}$$

$$V'_2(\hat{X}, \hat{Y}) = \frac{i}{\sqrt{2m}}[(\hat{P}_X W_X)(\hat{X}, \hat{Y}) - (\hat{P}_Y W_Y)(\hat{X}, \hat{Y})] + (W_X(\hat{X}, \hat{Y}))^2 + (W_Y(\hat{X}, \hat{Y}))^2, \tag{17}$$

$$V_2(\hat{X}, \hat{Y}) = \frac{-i}{\sqrt{2m}}[(\hat{P}_X W_X)(\hat{X}, \hat{Y}) + (\hat{P}_Y W_Y)(\hat{X}, \hat{Y})] + (W_X(\hat{X}, \hat{Y}))^2 + (W_Y(\hat{X}, \hat{Y}))^2. \tag{18}$$

Here V'_1, V'_2 and V_2 are, respectively, associated with H'_{11}, H'_{22} and H_2. We emphasize that this factorization method is more general than the direct noncommutative extension of the two-dimensional commutative supersymmetric formulation of [15]. Indeed, in the latter a unique superpotential $W(\hat{X}, \hat{Y})$ is required and raising and lowering operators are defined by substituting in (8) and (9) $W_X(\hat{X}, \hat{Y}) = (\hat{P}_X W)(\hat{X}, \hat{Y})$ and $W_Y(\hat{X}, \hat{Y}) = (\hat{P}_Y W)(\hat{X}, \hat{Y})$. The two-dimensional Riccati equation in the variables $(\hat{P}_X W, \hat{P}_Y W)$ can then be written in terms of a generalized noncommutative gradient of $\hat{P}_{X,Y} W$, namely

$$V_1(\hat{X}, \hat{Y}) = \frac{i}{\sqrt{2m}}\big(\hat{P}_X^2 W + \hat{P}_Y^2 W\big)(\hat{X}, \hat{Y}) + ((\hat{P}_X W)^2 + (\hat{P}_Y W)^2)(\hat{X}, \hat{Y}). \tag{19}$$

The natural question of the relation between energy eigenvalues and eigenfunctions of the different Hamiltonians H_1, H'_1, H'_2 and H_2 can now be considered, i.e., the supercharge formulation should be investigated. The following relations hold

$$H_1 B_X^\dagger = B_X^\dagger H'_{11} + B_Y^\dagger H'_{21} + \big[B_Y^\dagger, B_X^\dagger\big]B_Y, \tag{20}$$

$$H_1 B_Y^\dagger = B_X^\dagger H'_{12} + B_Y^\dagger H'_{22} - \big[B_Y^\dagger, B_X^\dagger\big]B_X, \tag{21}$$

$$B_X^\dagger H_2 = H'_{22} B_X^\dagger - H'_{21} B_Y^\dagger - B_Y\big[B_Y^\dagger, B_X^\dagger\big], \tag{22}$$

$$B_Y^\dagger H_2 = -H'_{12} B_X^\dagger + H'_{11} B_Y^\dagger + B_X\big[B_Y^\dagger, B_X^\dagger\big]. \tag{23}$$

Other types of relations can be obtained by taking the adjoint of these identities. Note that the last terms in (20)–(23) involve the commutator $\big[B_Y^\dagger, B_X^\dagger\big]$, which cannot vanish

without further assumptions. It turns out that, choosing $W_X(\hat{X}, \hat{Y}) = (\hat{P}_X W)(\hat{X}, \hat{Y})$ and $W_Y(\hat{X}, \hat{Y}) = (\hat{P}_Y W)(\hat{X}, \hat{Y})$, this commutator reduces to

$$\left[B_Y^\dagger, B_X^\dagger\right] = [\hat{P}_Y W, \hat{P}_X W], \tag{24}$$

and that in the commutative limit one recovers the two-dimensional supersymmetry as discussed in [15]. This underlines also the plausible scenario that noncommutativity could break the supersymmetry. Another issue that we shall not pursue here, but that could be of major interest, is the notion of noncommutative shape invariance, solvable potentials and Hamiltonian hierarchy [9]. As these concepts rest on algebraic commutation relations and the noncommutativity involves the adjoint action, it should be possible to investigate these concepts in the noncommutative setting. However, due to the dimension greater than one, these notions are not even yet well understood in the commutative case [18].

3.2. Application to the noncommutative harmonic oscillator

As an application of the above formalism, let us consider the noncommutative harmonic oscillator. The quantum Hamiltonian defining the motion of a non-relativistic particle of mass m confined in a harmonic well with frequency ω within a noncommutative plane can be written up to a constant[4] as

$$H_1 = \frac{1}{2m}\left(\hat{P}_X^2 + \hat{P}_Y^2\right) + \frac{1}{2}m\omega^2(\hat{X}^2 + \hat{Y}^2) - \hbar\omega. \tag{25}$$

The following superpotentials

$$W_X(\hat{X}, \hat{Y}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{X}, \qquad W_Y(\hat{X}, \hat{Y}) = \sqrt{\frac{m\omega}{2\hbar}}\hat{Y} \tag{26}$$

allow the following definition of the operators:

$$B_X = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} + \frac{\mathrm{i}}{m\omega}\hat{P}_X\right), \qquad B_X^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{X} - \frac{\mathrm{i}}{m\omega}\hat{P}_X\right), \tag{27}$$

$$B_Y = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{Y} + \frac{\mathrm{i}}{m\omega}\hat{P}_Y\right), \qquad B_Y^\dagger = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{Y} - \frac{\mathrm{i}}{m\omega}\hat{P}_Y\right). \tag{28}$$

These operators admit the following factorization of the Hamiltonian H_1:

$$H_1 = \hbar\omega\left(B_X^\dagger B_X + B_Y^\dagger B_Y\right). \tag{29}$$

The associated Hamiltonians are easily computed

$$H_2 = H_1 + 2\hbar\omega\mathbb{I}, \qquad H'_{11} = H_1 + \hbar\omega = H'_{22}, \qquad H'_{12} = -\frac{\mathrm{i}m\omega^2\theta}{2} = -H'_{21}. \tag{30}$$

The noncommutative supersymmetric factorization can therefore be carried out exactly for the noncommutative harmonic oscillator. However, the quantum supercharges cannot be identified with these operators. Indeed, we can check that $\left[B_X^\dagger, B_Y^\dagger\right] \propto \theta$ implies that (20)–(23) do not define a superalgebra. It is only through a re-factorization of the Hamiltonian with respect to decoupled operators that the superalgebra emerges.

To proceed, we write H_1 (29) in the matrix form

$$H = \hbar\omega B^\dagger B, \qquad B = (B_X, B_Y)^t, \tag{31}$$

---

[4] This constant, $-\hbar\omega$, has to be related to the unbroken supersymmetry as appears hereafter.

where symbol t denotes the transpose operation. The purpose of this rewriting is the factorization of the Hamiltonian in terms of diagonal bosonic operators:

$$H = \hbar\omega A^\dagger D A, \qquad A = (A_+, A_-)^t, \tag{32}$$

where D is some diagonal positive matrix and $\left(A_\pm, A_\pm^\dagger\right)$ satisfy decoupled and diagonal bosonic commutation relations,

$$\left[A_\pm, A_\pm^\dagger\right] = \mathbb{I}. \tag{33}$$

We introduce the vectors $A^+ = \left(A_+^\dagger, A_-^\dagger\right)^t$ and $B^+ = \left(B_X^\dagger, B_Y^\dagger\right)^t$, for which $(A^+)^t = A^\dagger$, as well as a linear transformation S relating them

$$A = SB, \qquad A^+ = S^* B^+. \tag{34}$$

Defining the matrix $\mathfrak{g}$ with elements

$$\mathfrak{g}_{kl} = \left[B_k, B_l^+\right], \qquad k, l = 1, 2, \qquad B_1 := B_X, \qquad B_2 := B_Y, \tag{35}$$

it is simple to verify that $\mathfrak{g}$ is Hermitian. Let us denote its eigenvectors by u_i, $i = 1, 2$. From the commutation relations

$$\left[A_i, A_j^+\right] = \delta_{ij}, \qquad i, j = 1, 2, \tag{36}$$

one derives the identity

$$S\mathfrak{g}S^\dagger = \mathbb{I}_2. \tag{37}$$

The Hamiltonian diagonalization is now immediate with the following choice of the matrix $S^\dagger = (u_1, u_2)$. Noting from (37) that $(S^\dagger)^{-1} = S\mathfrak{g}$, we get from (34), when inserted into (31), the following result

$$H = \hbar\omega (A^+)^t S\mathfrak{g}^2 S^\dagger A. \tag{38}$$

It simply remains to apply $\mathfrak{g}$ twice on its eigenstates to obtain the diagonal form of the Hamiltonian. The eigenvectors have not yet been normalized. The normalization conditions are fixed from the requirement that for $i = 1, 2$, $(u_i)^\dagger u_i = 1/|\lambda_i|$, where λ_i is the eigenvalue associated with u_i. Thus, in terms of these boson operators the Hamiltonian can be expressed as

$$H = \hbar\omega\left(|\lambda_+|A_+^\dagger A_+ + |\lambda_-|A_-^\dagger A_-\right), \tag{39}$$

where $|\lambda_\pm|$ are the absolute values of the eigenvalues of the matrix

$$\mathfrak{g} = \begin{pmatrix} 1 & \mathbf{a}_\theta \\ -\mathbf{a}_\theta & 1 \end{pmatrix}. \tag{40}$$

Here we have introduced the parameter $\mathbf{a}_\theta = \frac{1}{2\hbar} m\omega\theta$. The computation of these eigenvalues is easily performed with the result

$$|\lambda_+| = 1 + \mathbf{a}_\theta, \qquad |\lambda_-| = |1 - \mathbf{a}_\theta|, \tag{41}$$

which diagonalizes the Hamiltonian (31). This operator is nothing but the Hamiltonian describing a harmonic oscillator with frequency encoding the noncommutative parameters $(\theta, \hbar)$. In terms of B these bosonic operators can be expressed as

$$A_+ = \frac{1}{c_+}(-iB_X + B_Y), \qquad A_- = \frac{1}{c_-}(iB_X + B_Y),$$
$$c_+ = \sqrt{2(1 + \mathbf{a}_\theta)}, \qquad c_- = \sqrt{2|1 - \mathbf{a}_\theta|}, \tag{42}$$

while $A_\pm^\dagger$ are obtained by the adjoint. Finally, we mention that this factorization still works for a model with broken rotational symmetry resulting from different frequencies (ω_X, ω_Y) for

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the noncommutative directions. In this case, one considers, without loss of generality, scaled operators in one direction, for instance $\hat{Y}$, namely, $\widetilde{B}_Y = \sqrt{(\omega_Y/\omega_Y)}B_Y$, and proceeds in the same way as before. One arrives at the following factorized operator:

$$H' = \hbar\left(|\lambda'_+|A'^\dagger_+A'_+ + |\lambda'_-|A'^\dagger_-A'_-\right),$$

$$|\lambda'_\pm| = \left|\frac{\omega_X + \omega_Y}{2} \pm \frac{1}{2}\sqrt{(\omega_X - \omega_Y)^2 + \left(\frac{m\omega_X\omega_Y\theta}{\hbar}\right)^2}\right|, \tag{43}$$

which, of course reduces to (39) when $\omega_X = \omega_Y = \omega$.

At this point, we are able to identify the quantum supercharges. Fixing the constant parameters $\kappa_\pm = \sqrt{\hbar\omega|\lambda_\pm|}$, the following operator

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_+ A_+ & 0 & 0 & 0 \\ \kappa_- A_- & 0 & 0 & 0 \\ 0 & \kappa_- A_- & -\kappa_+ A_+ & 0 \end{pmatrix} \tag{44}$$

and its adjoint $Q^\dagger$ satisfy a $\mathcal{N} = 2$ superalgebra such that

$$\{Q, Q^\dagger\} = \mathcal{H}, \qquad [Q, \mathcal{H}] = 0 = [Q^\dagger, \mathcal{H}], \tag{45}$$

with the 4×4 matrix Hamiltonian given by

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}^1_+ + \mathcal{H}^1_- & 0 & 0 & 0 \\ 0 & \mathcal{H}^2_+ + \mathcal{H}^1_- & 0 & 0 \\ 0 & 0 & \mathcal{H}^2_- + \mathcal{H}^1_+ & 0 \\ 0 & 0 & 0 & \mathcal{H}^2_+ + \mathcal{H}^2_- \end{pmatrix}, \tag{46}$$

$$\mathcal{H}^1_\pm = \hbar\omega|\lambda_\pm|A^\dagger_\pm A_\pm, \qquad \mathcal{H}^2_\pm = \hbar\omega|\lambda_\pm|A_\pm A^\dagger_\pm. \tag{47}$$

We get the following relations between the Hamiltonians:

$$\mathcal{H}_{11}(\kappa_+ A^\dagger_+) = (\kappa_+ A^\dagger_+)\mathcal{H}_{22}, \qquad \mathcal{H}_{11}(\kappa_+ A^\dagger_-) = (\kappa_+ A^\dagger_-)\mathcal{H}_{33}, \tag{48}$$

$$(\kappa_+ A^\dagger_+)\mathcal{H}_{44} = \mathcal{H}_{33}(\kappa_+ A^\dagger_+), \qquad (\kappa_+ A^\dagger_-)\mathcal{H}_{44} = \mathcal{H}_{22}(\kappa_+ A^\dagger_+), \tag{49}$$

$$\mathcal{H}_{11} = \mathcal{H}^1_+ + \mathcal{H}^1_-, \qquad \mathcal{H}_{22} = \mathcal{H}^2_+ + \mathcal{H}^1_-, \qquad \mathcal{H}_{33} = \mathcal{H}^1_+ + \mathcal{H}^2_-, \qquad \mathcal{H}_{44} = \mathcal{H}^2_+ + \mathcal{H}^2_-, \tag{50}$$

which are to be compared with (20)–(23). From this one can obtain the correct relations between the operator eigenfunctions of these Hamiltonians.

Each of the matrix operators Q and $Q^\dagger$ can indeed be decomposed into four elementary matrices labeled by their entries and each of these 'sub-operators' represents a symmetry of the Hamiltonian. In that language, we should say that the model is $\mathcal{N} = 8$ supersymmetric. Nevertheless, as a matter of compact notation we keep to the operators Q and $Q^\dagger$, which generate a $\mathcal{N} = 2$ supersymmetry. It should be emphasized that in the commutative limit $\theta \to 0$ the operator

$$Q' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_- A_- & 0 & 0 & 0 \\ \kappa_+ A_+ & 0 & 0 & 0 \\ 0 & \kappa_+ A_+ & -\kappa_- A_- & 0 \end{pmatrix} \tag{51}$$

and its adjoint $Q'^\dagger$ also provide a set of symmetries for the Hamiltonian. In the presence of noncommutative geometry, i.e., when the noncommutative parameter $\theta \neq 0$, the total supersymmetry is partially broken and the number of supercharges decreases from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ (actually from $\mathcal{N} = 16$ to $\mathcal{N} = 8$). Finally, let us mention that this partial supersymmetry breaking is in agreement with the supersymmetry reduction $\mathcal{N} = 1 \rightarrow 1/2$ as pointed out by Seiberg in the context of noncommutative superfield theory in four-dimensional noncommutative spacetime [2].

Let us now investigate the relationship between this partial supersymmetry breaking and time reversal asymmetry arising from noncommutativity. We start by calculating the total angular momentum operator. Given the tensor product supercharges

$$Q_1 = \kappa_+ A_+ \sigma_- \otimes \sigma_3, \qquad Q_1^\dagger = \kappa_+ A_+^\dagger \sigma_+ \otimes \sigma_3, \qquad (52)$$

$$Q_2 = \kappa_- A_- \mathbb{I}_2 \otimes \sigma_-, \qquad Q_2^\dagger = \kappa_- A_-^\dagger \mathbb{I}_2 \otimes \sigma_+, \qquad (53)$$

we can write

$$Q = Q_1 + Q_2, \qquad Q^\dagger = Q_1^\dagger + Q_2^\dagger. \qquad (54)$$

The Hamiltonian can then be written as

$$\mathcal{H} = \left\{Q_1, Q_1^\dagger\right\} + \left\{Q_2, Q_2^\dagger\right\} = \mathcal{H}' \mathbb{I}_2 \otimes \mathbb{I}_2 - \frac{\kappa_+^2}{2}\sigma_3 \otimes \mathbb{I}_2 - \frac{\kappa_-^2}{2}\mathbb{I}_2 \otimes \sigma_3,$$

$$\mathcal{H}' = \kappa_+^2 A_+^\dagger A_+ + \kappa_-^2 A_-^\dagger A_- + \frac{\kappa_+^2}{2} + \frac{\kappa_-^2}{2}. \qquad (55)$$

The system therefore admits a total spin operator as

$$S = \frac{\hbar}{2}(-\mathbb{I}_2 \otimes \sigma_3 + \sigma_3 \otimes \mathbb{I}_2). \qquad (56)$$

The following identities can be deduced

$$[S, \sigma_- \otimes \sigma_3] = -\hbar \sigma_- \otimes \sigma_3 \qquad [S, \mathbb{I}_2 \otimes \sigma_-] = \hbar \mathbb{I}_2 \otimes \sigma_-. \qquad (57)$$

The noncommutative orbital angular momentum operator L_z is given by [7]

$$L_z = \hat{X}\hat{P}_Y - \hat{Y}\hat{P}_X + \frac{\theta}{2\hbar}\left(\hat{P}_X^2 + \hat{P}_Y^2\right) \qquad (58)$$

and satisfies the algebra $[L_z, \hat{X}] = i\hbar \hat{Y}$ and $[L_z, \hat{Y}] = -i\hbar \hat{X}$. Using the inverse relations (42), the following commutation relations can be obtained

$$[L_z, A_\pm] = \pm\hbar A_\pm, \qquad \left[L_z, A_\pm^\dagger\right] = \mp\hbar A_\pm^\dagger. \qquad (59)$$

Finally, the total angular momentum operator is

$$J = L_z \mathbb{I}_4 + S. \qquad (60)$$

Using (57) and (59), the supercharge Q transforms as follows:

$$[J, Q] = \kappa_+([L_z, A_+] - \hbar A_+)\sigma_- \otimes \sigma_3 + \kappa_-([L_z, A_-] + \hbar A_-)\mathbb{I}_2 \otimes \sigma_- = 0. \qquad (61)$$

Therefore, the supercharge Q and the angular momentum L_z are commuting objects. On the other hand, one can immediately check that $[L_z, Q'] \neq 0$ since by switching the operators $A_+ \rightarrow A_-$, this commutator becomes nontrivial. As breaking of time reversal symmetry lifts the degeneracy between states with angular momentum $+m$ and $-m$, it is clear that only the supercharge Q that does not change the angular momentum can survive in the noncommutative limit. Indeed, it is not difficult to check that this is exactly half of the supercharges in the commutative case.

4. BPS states

BPS states saturate the lower energy bound in a given charge sector [16, 19]. They play an important role in the analysis of many systems as they satisfy simpler equations, which often allows the exact computation of their spectrum. In supersymmetric systems they generically possess only a fraction of the full supersymmetry, 1/2 BPS states, which possess only half of the supersymmetry, being a typical example. Due to the importance of these states, it is useful to clarify their role in the present setting.

The first point to realize is that the supersymmetry breaking arising from the noncommutativity of space is not related to the supersymmetry breaking occurring in BPS states. Indeed, in the latter case the Hamiltonian still has the full supersymmetry, it is the states that have a reduced supersymmetry. The supersymmetry breaking discussed in the previous sections is an explicit breaking of supersymmetry in the sense that half of the supercharges are no longer conserved in the presence of noncommutativity. In the present setting it is quite straightforward to identify the BPS states and their supersymmetry. First note that the Hilbert space can be written as the direct sum of sectors with fixed angular momentum. In each sector we can then identify those states that give the lowest possible energy. It is simple to see that for negative angular momentum these are the states

$$\begin{pmatrix} |n_+, 0\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ |n_+ - 1, 0\rangle \\ 0 \\ 0 \end{pmatrix}. \tag{62}$$

These states have angular momentum $m = -n_+$ and energy $E = -m$, which is indeed the lowest energy bound in this sector. For positive angular momentum the corresponding states are

$$\begin{pmatrix} |0, n_-\rangle \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0 \\ |0, n_- - 1\rangle \\ 0 \end{pmatrix}. \tag{63}$$

These states have angular momentum $m = n_-$ and energy $E = m$, which is again the lowest energy bound in this sector.

The pairs of states (62) and (63) are BPS states, related by a time reversal transformation. In the noncommutative case each pair is degenerate, while the two pairs are nondegenerate and each is annihilated by six of the conserved eight supercharges (44). Note, however, that these six supercharges are different for the positive and negative angular momentum pairs. Each pair therefore represents a 3/4 BPS doublet in which the remaining two supercharges ladder between the two members of the pair. In the commutative case all angular momentum states and in particular these pairs are degenerate. Due to this degeneracy there are eight more conserved supercharges (51) and each pair forms part of a 1/2 BPS octet. This octet gets broken pairwise when noncommutativity is switched on. It is worthwhile noting that in the $m = 0$ sector, the lowest energy state is the ground state of the system which has the full supersymmetry, i.e., eight in the noncommutative and sixteen in the commutative case.

5. Conclusion

We have extended the fundamental notions of supersymmetric factorization to two-dimensional noncommutative quantum systems. The new approach was then applied to factorize the

noncommutative harmonic oscillator. The diagonalization of the quantum Hamiltonian operator has been successfully performed and, consequently, we determined the quantum supercharges. It turns out that the supersymmetry is partially broken and the number of supercharges decreases in the presence of noncommutativity. Furthermore, we have discussed how this supersymmetry breaking is related to time reversal symmetry breaking. Finally, in the commutative limit the usual properties are recovered.

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